

Essential Signatures and Canonical Bases for B_n and D_n .

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Abstract

In the representation theory of simple Lie algebras, we consider the problem of constructing a “canonical” weight basis in an arbitrary irreducible finite-dimensional highest weight module. Vinberg suggested a method for constructing such bases by applying the lowering operators corresponding to all positive roots to the highest weight vector. These “canonical” bases were constructed for the cases A_n , C_n , G_2 , B_3 , D_4 . In this paper, we construct such bases for Lie algebras of types B_n and D_n .

1 Introduction

Let \mathfrak{g} be a simple Lie algebra. One has the triangular decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$, where \mathfrak{u}^- and \mathfrak{u} are mutually opposite maximal unipotent subalgebras and \mathfrak{t} is a Cartan subalgebra.

One has: $\mathfrak{u} = \langle e_\alpha \mid \alpha \in \Delta_+ \rangle$, $\mathfrak{u}^- = \langle e_{-\alpha} \mid \alpha \in \Delta_+ \rangle$, where Δ_+ is the system of positive roots, $e_{\pm\alpha}$ are the root vectors, and the symbol $\langle \dots \rangle$ stands for the linear span.

We denote a finite-dimensional irreducible \mathfrak{g} -module with highest weight λ by $V(\lambda)$ and a highest weight vector in this module by v_λ .

Various approaches to construction of canonical bases in $V(\lambda)$ are known: Gelfand–Tsetlin bases, crystal bases, etc. For instance, one constructs the crystal basis by applying the lowering operators corresponding to simple roots to the highest weight vector in a certain order; see [7]. Vinberg’s method [10] resembles the method for constructing crystal bases; the main difference is that the lowering operators corresponding to *all positive roots* are applied to the highest weight vector. Vinberg’s and crystal bases can be obtained by a

uniform procedure (see [8] for details). The basic concept used in Vinberg's method is defined as follows.

Definition 1. A signature is an $(N + 1)$ -tuple $\sigma = (\lambda; p_1, \dots, p_N)$, where N is the number of positive roots numbered in a certain fixed order: $\Delta_+ = \{\alpha_1, \dots, \alpha_N\}$, λ is a dominant weight, and $p_i \in \mathbb{Z}_+$.

Set

$$v(\sigma) = e_{-\alpha_1}^{p_1} \cdot \dots \cdot e_{-\alpha_N}^{p_N} \cdot v_\lambda.$$

λ is called the *highest weight* of σ and the weight $\lambda - \sum p_i \alpha_i$ of the vector $v(\sigma)$ is called the *weight* of σ . Thus we have defined a vector in $V(\lambda)$ for every signature with highest weight λ . The vectors $v(\sigma)$ generate $V(\lambda)$, but they are linearly dependent. Our goal is to select a basis of $V(\lambda)$ from the set of all vectors $v(\sigma)$.

This problem was solved in [2], [3], [4], [1], [5] for the algebras of types A_n , C_n , G_2 , B_3 , and D_4 , respectively. Let $\mathfrak{t}_\mathbb{Z} \subset \mathfrak{t}$ be the coroot lattice, i.e., the lattice of vectors on which all weights take integer values. The signatures corresponding to the basis vectors are determined by a set of linear inequalities of the form

$$\sum_{j \in M_i} a_{ij} p_j \leq \lambda(l_i), \tag{1}$$

where $M_i \subset \{1, \dots, N\}$ are certain subsets, $l_i \in \mathfrak{t}_\mathbb{Z}$, $a_{ij} = 1$ or 2 in type B_3 , $a_{ij} = 1, 2$ or 3 in type D_4 , and $a_{ij} = 1$ otherwise.

The signatures corresponding to basis vectors of all irreducible \mathfrak{g} -modules form an additive semigroup in all these cases. Moreover, this semigroup is generated by the signatures of fundamental highest weights. We want to obtain the same result for Lie algebras of types B_n and D_n , but in these cases the semigroup is generated by signatures of fundamental highest weights and some other highest weights.

Now we explain our approach to solving the problem. To this end, we need to equip the set of signatures with an order. Let $\omega_1, \dots, \omega_n$ be the fundamental weights and let

$$\begin{aligned} \sigma &= (\lambda; p_1, \dots, p_N), \quad \sigma' = (\lambda'; p'_1, \dots, p'_N), \\ \lambda &= \sum k_i \omega_i, \quad \lambda' = \sum k'_i \omega_i, \quad k_i, k'_i \in \mathbb{Z}_+. \end{aligned}$$

First we compare the tuples (k_1, \dots, k_n) and (k'_1, \dots, k'_n) by using the degree lexicographic order and put $\sigma < \sigma'$ if $(k_1, \dots, k_n) < (k'_1, \dots, k'_n)$. If $\lambda = \lambda'$, then we compare the tuples (p_1, \dots, p_N) and (p'_1, \dots, p'_N) by using a fixed monomial order on $\mathbb{Z}_{\geq 0}^N$.

Definition 2. *A signature σ is essential, if $v(\sigma) \notin \langle v(\tau) \mid \tau < \sigma \rangle$.*

The following statement is obvious.

Proposition 1. *The set $\{v(\sigma) \mid \sigma \text{ essential}\}$ is a basis of $V(\lambda)$.*

The essential signatures with given highest weight λ parametrize the desired canonical basis of $V(\lambda)$. The following proposition was proved by Vinberg. For convenience of the reader, we provide a proof in Section 2:

Proposition 2. *The essential signatures form a semigroup Σ in $\mathfrak{t}_{\mathbb{Z}}^* \oplus \mathbb{Z}^N$.*

Now we proceed to the first conjecture of Vinberg about the structure of the set of essential signatures.

Conjecture 1. *The semigroup Σ is generated by the essential signatures of fundamental highest weights.*

Let us formulate other conjectures of Vinberg. Let $\Sigma_{\mathbb{Q}}$ be the rational cone spanned by Σ . Then this cone can be defined by linear inequalities. (The number of these inequalities is finite if Conjecture 1 holds.)

Conjecture 2. *The semigroup Σ is saturated, i.e., $\Sigma = \Sigma_{\mathbb{Q}} \cap (\mathfrak{t}_{\mathbb{Z}}^* \oplus \mathbb{Z}^N)$.*

Conjecture 2 claims that the bases of $V(\lambda)$ are parametrized by lattice points of plane sections of some polyhedral cone.

Conjecture 3. *There exist a family of subsets $M_i \subset \{1, \dots, N\}$ and a family of elements $l_i \in \mathfrak{t}_{\mathbb{Z}}$ such that the set of essential signatures $\sigma = (\lambda; p_1, \dots, p_N)$ of highest weight λ is given by the inequalities*

$$\sum_{j \in M_i} p_j \leq \lambda(l_i).$$

Conjecture 3 refines the structure of the polyhedral cone in Conjecture 2.

Conjectures 1, 2, 3 were proved in cases A_n , C_n and G_2 ([2], [3], [4]). Conjecture 1, 2 and modified version of Conjecture 3 were proved in cases B_3 , D_4 ([1], [5]).

In Section 3 we give a necessary and sufficient condition for Conjecture 1 to be true, and we explain how this condition can be verified. In the rest of the article we construct “canonical” bases for B_n and D_n by using inductive procedure, starting from $D_3 = A_3$. We prove Conjecture 2 and the modified version of Conjecture 1 for B_n and D_n .

2 The semigroup of essential signatures

Here we show that the essential signatures of all highest weights form a semigroup.

Let G be a simply connected simple complex algebraic group such that $\text{Lie } G = \mathfrak{g}$. Let T be the maximal torus in G such that $\text{Lie } T = \mathfrak{t}$ and U be the maximal unipotent subgroup of G such that $\text{Lie } U = \mathfrak{u}$. Consider the homogeneous space G/U . Let $B = T \ltimes U$ be the Borel subgroup. Then

$$\mathbb{C}[G/U] = \bigoplus_{\lambda} \mathbb{C}[G]_{\lambda}^{(B)},$$

where

$$\mathbb{C}[G]_{\lambda}^{(B)} = \{f \in \mathbb{C}[G] \mid f(gtu) = \lambda(t)f(g), \forall g \in G, t \in T, u \in U\}$$

is the subspace of eigenfunctions of weight λ for B acting on $\mathbb{C}[G]$ by right translations of an argument. Each subspace $\mathbb{C}[G]_{\lambda}^{(B)}$ is finite-dimensional and is isomorphic as a G -module (with respect to the action of G by left translations of an argument) to the space $V(\lambda)^*$ of linear functions on $V(\lambda)$ (see, e.g., [9], Theorem 3). The isomorphism is given by the formula:

$$V(\lambda)^* \ni \omega \mapsto f_{\omega} \in \mathbb{C}[G]_{\lambda}^{(B)}, \quad \text{where} \quad f_{\omega}(g) = \langle \omega, gv_{\lambda} \rangle.$$

Let U^- be the maximal unipotent subgroup such that $\text{Lie } U^- = \mathfrak{u}^-$. The function f_{ω} is uniquely determined by its restriction to the dense open subset $U^- \cdot T \cdot U$; moreover

$$f_{\omega}(u^- \cdot t \cdot u) = \langle \omega, u^- t u v_{\lambda} \rangle = \langle \omega, \lambda(t) u^- v_{\lambda} \rangle = \lambda(t) f_{\omega}(u^-), \\ \forall u \in U, u^- \in U^-, t \in T.$$

Next, $U^- = U_{-\alpha_1} \cdot \dots \cdot U_{-\alpha_N}$, where $U_\alpha = \{\exp(ze_\alpha) \mid z \in \mathbb{C}\}$ (see [6, Sec. X, §28.1]). Hence

$$u^- = \exp(z_1 e_{-\alpha_1}) \cdot \dots \cdot \exp(z_N e_{-\alpha_N}).$$

Thus we obtain

$$f_\omega(u^-) = \langle \omega, \exp(z_1 e_{-\alpha_1}) \cdot \dots \cdot \exp(z_N e_{-\alpha_N}) \cdot v_\lambda \rangle = \sum_{\sigma=(\lambda; p_1, \dots, p_N)} \frac{\prod z_i^{p_i}}{\prod p_i!} \langle \omega, v(\sigma) \rangle.$$

Proposition 3. *A signature σ is essential if and only if $\prod z_i^{p_i}$ is the least term in $f_\omega|_{U^-}$ for some $\omega \in V(\lambda)^*$ in the sense of the order introduced above.*

Proof. Let $\prod z_i^{p_i}$ be the least term in $f_\omega|_{U^-}$ for some $\omega \in V(\lambda)^*$. Then ω vanishes on all vectors $v(\tau)$ with $\tau < \sigma$ and is nonzero at $v(\sigma)$. Consequently, $v(\sigma)$ cannot be expressed via $v(\tau)$ with $\tau < \sigma$, and hence σ is essential.

Conversely, let σ be essential. Consider a function ω that vanishes on $v(\tau)$ for all essential τ except σ . Obviously, $f_\omega|_{U^-}$ has the desired least term. \square

Proof of Proposition 2. Suppose that the least terms in $f|_{U^-}$ and $g|_{U^-}$ correspond to the essential signatures σ and π of highest weights λ and μ , respectively. Then the least term in $(f \cdot g)|_{U^-}$ corresponds to the signature $\sigma + \pi$ of highest weight $\lambda + \mu$. Hence $\sigma + \pi$ is essential. \square

3 Conjecture 1

Here we give a necessary and sufficient condition for Conjecture 1 to be true.

It is known that the algebra $\mathbb{C}[G/U]$ is generated by the subspaces $\mathbb{C}[G/U]_{\omega_i}^{(B)}$ ($i = 1, \dots, n$). Set $X = \text{Spec } \mathbb{C}[G/U]$. Then

$$X \simeq \overline{G(v_{\omega_1} + \dots + v_{\omega_n})} \subseteq V(\omega_1) \oplus \dots \oplus V(\omega_n) = V.$$

Let I be the vanishing ideal of X in $\mathbb{C}[V]$, and x_i be the coordinates on $V(\omega_1) \oplus \dots \oplus V(\omega_n)$ corresponding to the basis $\{v(\sigma_i) \mid \sigma_i \text{ essential}\}$. So we have a surjective algebra homomorphism

$$\phi: \mathbb{C}[V] \rightarrow \mathbb{C}[X],$$

such that $\ker(\phi) = I$.

Consider an arbitrary monomial $x_{i_1} \dots x_{i_k}$ in $\mathbb{C}[V]$. Set $\text{sign}(x_{i_1} \dots x_{i_k}) = \sigma_{i_1} + \dots + \sigma_{i_k} = \sigma$. We call σ *the signature of the monomial* $x_{i_1} \dots x_{i_k}$. This defines a grading of $\mathbb{C}[V]$ by the subsemigroup of Σ generated by essential signatures of fundamental highest weights. The degree of any polynomial $h \in \mathbb{C}[V]$ homogeneous with respect to this grading is also denoted by $\text{sign}(h)$. For any $f \in \mathbb{C}[V]$, denote by $\text{int}(f)$ the least term of f with respect to this grading. Let $\text{int}(I)$ be the ideal spanned by $\text{int}(f), f \in I$. Denote by J the ideal spanned by the binomials

$$x_{i_1} \dots x_{i_k} - x_{j_1} \dots x_{j_k} \quad \text{such that} \quad \text{sign}(x_{i_1} \dots x_{i_k}) = \text{sign}(x_{j_1} \dots x_{j_k}).$$

Let f be a regular function on V , which can be expressed as a polynomial in $\{x_i\}$. Then $\phi(f)$ is a function on X , hence a polynomial in $\{z_i\}, i = 1, \dots, N$ (coordinates on U^-), and $\{t_j\}, j = 1, \dots, n$ (coordinates on T corresponding to fundamental weights). Thus if x_i is the coordinate function corresponding to the signature $\sigma_i = (\omega_i; p_1, \dots, p_N)$, then $\phi(x_i)$ is the polynomial in $\{z_i\}$ and $\{t_j\}$ with the least term $t_l \prod z_k^{p_k}$.

Lemma 1. $\text{int}(I) \subset J$.

Proof. Let $f \in I$, then $\phi(f) = 0$. Let $\sigma = (\lambda; p_1, \dots, p_N) = \text{sign}(\text{int}(f))$. The least term of $\phi(\text{int}(f))$ is greater than $t^\lambda \prod z_k^{p_k}$, because $\phi(f) = 0$. Hence $\text{int}(f)$ is a linear combination of monomials of signature σ with the sum of coefficients equal to 0, i.e. $\text{int}(f) \in J$. \square

Proposition 4. *Conjecture 1 is true if and only if $\text{int}(I) = J$.*

Proof. Conjecture 1 claims that for any polynomial $g \in \mathbb{C}[V]$ the least term of $\phi(g)$ coincides with the least term of some $\phi(x_{i_1} \dots x_{i_k})$. Assume that Conjecture 1 is true and let $f \in J$. We can suppose that f is homogeneous for the signature grading. Let $\sigma = (\lambda; p_1, \dots, p_N) = \text{sign}(f)$. Then the least term of $\phi(f)$ is greater than $t^\lambda \prod z_k^{p_k}$. There exists a monomial $x_{i_1} \dots x_{i_k}$ such that $\phi(x_{i_1} \dots x_{i_k})$ has the same least term as $\phi(f)$. So, for some constant c the least term of $\phi(f_1)$, where $f_1 = f - cx_{i_1} \dots x_{i_k}$, is greater than the least term of $\phi(f)$. We can continue this process, subtracting the monomials from f_s to obtain f_{s+1} such that the least term of $\phi(f_{s+1})$ is greater than the least term of $\phi(f_s)$. The process will stop, since there exist finitely many essential signatures with fixed highest weight. Hence we obtain f_m such that $\phi(f_m) = 0$ and $\text{int}(f_m) = f$. Thus $f \in \text{int}(I)$.

Conversely, suppose that $\text{int}(I) = J$. Let $f \in \mathbb{C}[V]$. We look for a monomial $x_{i_1} \dots x_{i_k}$ such that the least term of $\phi(x_{i_1} \dots x_{i_k})$ coincides with the least term of $\phi(f)$. We can assume that f is a polynomial such that the least signature of a monomial in f is maximal over polynomials in $\phi^{-1}\phi(f)$. Let $h = \text{int}(f)$. If $h \notin J$, then we may take for $x_{i_1} \dots x_{i_k}$ any monomial of h . Otherwise $h \in J$, hence there exist $f' \in I$ such that $h = \text{int}(f')$. Then $\phi(f - f') = \phi(f)$ and the least signature of monomials in $f - f'$ is greater than in f . A contradiction. \square

Now we want to explain how the condition above can be verified. To prove the equality $\text{int}(I) = J$ it is enough to verify two properties:

1. the ideal J is generated by binomials of degree 2;
2. for any two fundamental weights ω_i, ω_j any essential signature of the highest weight $\omega_i + \omega_j$ is representable as a sum of essential signatures of fundamental highest weights.

Indeed, if the first property holds then to show that $\text{int}(I) = J$ it is enough to prove that if $f \in J$ is a polynomial of degree 2 then $f \in \text{int}(I)$. It follows from the first part of the proof of previous proposition ($\text{sign}(f)$ has the highest weight $\omega_i + \omega_j$), that it is true if and only if the second property holds.

Now we discuss how the first property can be verified. Let

$$\sigma = \sigma_1 + \dots + \sigma_k = \tau_1 + \dots + \tau_k,$$

where σ_i and τ_j are essential signatures of fundamental highest weights. Consider the following operation: replace a pair of signatures σ_i, σ_j by a pair of signatures σ'_i, σ'_j of fundamental highest weights such that $\sigma_i + \sigma_j = \sigma'_i + \sigma'_j$. We call this operation *admissible*. Obviously, the ideal J is generated by binomials of degree 2 if and only if for any such σ we can obtain a given decomposition $\sigma_1 + \dots + \sigma_k$ of σ from any other decomposition $\tau_1 + \dots + \tau_k$ by applying such operations.

The second property can be verified as follows. First, we compute the numbers of signatures of highest weights $\omega_i + \omega_j$, which can be represented as a sum of essential signatures of fundamental highest weights for all i, j . Then, we compare this number with $\dim V(\omega_i + \omega_j)$, which can be found by Weyl's dimension formula.

Denote by Σ^f the semigroup generated by essential signatures of fundamental highest weights. Above arguments allow us to reformulate Proposition 4:

Proposition 5. *Suppose that the following two properties hold:*

1. *for any signature $\sigma \in \Sigma^f$ of highest weight λ we can obtain a given decomposition $\sigma = \sigma_1 + \dots + \sigma_k$ into a sum of essential signatures of fundamental highest weights from any other such decomposition $\tau_1 + \dots + \tau_k$ by applying admissible operations;*
2. *for any two fundamental weights ω_i, ω_j any essential signature of highest weight $\omega_i + \omega_j$ is representable as a sum of essential signatures of highest weights ω_i and ω_j .*

Then any essential signature of highest weight λ is representable as a sum of essential signatures of highest weights $\{\omega_1, \dots, \omega_n\}$.

Generalization. Now we formulate a more general statement. Let $\lambda_1, \dots, \lambda_m$ be some dominant weights, σ be an essential signature of highest weight λ ,

$$\sigma = \sigma_1 + \dots + \sigma_k = \tau_1 + \dots + \tau_l$$

be two decompositions of σ , where σ_i and τ_j are essential signatures with highest weights in $\{\lambda_1, \dots, \lambda_m\}$. Define an admissible operation as either replacing a pair of signatures σ_i, σ_j of highest weights in $\{\lambda_1, \dots, \lambda_m\}$ by a pair of signatures σ'_i, σ'_j of highest weights in $\{\lambda_1, \dots, \lambda_m\}$ such that $\sigma_i + \sigma_j = \sigma'_i + \sigma'_j$, or replacing a pair of signatures σ_i, σ_j of highest weights in $\{\lambda_1, \dots, \lambda_m\}$ by a signature σ_m of highest weight in $\{\lambda_1, \dots, \lambda_m\}$ such that $\sigma_i + \sigma_j = \sigma_m$, or replacing a signature σ_m of highest weight in $\{\lambda_1, \dots, \lambda_m\}$ by a pair of signatures σ_i, σ_j of highest weights in $\{\lambda_1, \dots, \lambda_m\}$ such that $\sigma_m = \sigma_i + \sigma_j$.

Denote by $\Sigma^{(\lambda_1, \dots, \lambda_m)}$ the semigroup generated by essential signatures of highest weights in $\{\lambda_1, \dots, \lambda_m\}$. We have the following statement:

Theorem 1. *Suppose that the following two properties hold:*

- * *for any signature $\sigma \in \Sigma^{(\lambda_1, \dots, \lambda_m)}$ of highest weight λ we can obtain a given decomposition $\sigma = \sigma_1 + \dots + \sigma_k$ into a sum of essential signatures of highest weights in $\{\lambda_1, \dots, \lambda_m\}$ from any other such decomposition $\tau_1 + \dots + \tau_l$ by applying admissible operations;*

*** for any two weights λ_i, λ_j any essential signature of highest weight $\lambda_i + \lambda_j$ is representable as a sum of essential signatures of highest weights in $\{\lambda_1, \dots, \lambda_m\}$.*

Then any essential signature of highest weight λ is representable as a sum of essential signatures of highest weights in $\{\lambda_1, \dots, \lambda_m\}$.

Proof. First of all we make some modification of notations.

$$X \simeq \overline{G(v_{\lambda_1} + \dots + v_{\lambda_m})} \subseteq V(\lambda_1) \oplus \dots \oplus V(\lambda_m) = V.$$

Let I be the vanishing ideal of X in $\mathbb{C}[V]$, and x_i be the coordinates on $V(\lambda_1) \oplus \dots \oplus V(\lambda_m)$ corresponding to the basis $\{v(\sigma_i) \mid \sigma_i \text{ essential}\}$. We have a surjective algebra homomorphism

$$\phi: \mathbb{C}[V] \rightarrow \mathbb{C}[X],$$

such that $\ker(\phi) = I$.

Denote by J the ideal spanned by the binomials

$$x_{i_1} \dots x_{i_k} - x_{j_1} \dots x_{j_l} \quad \text{such that} \quad \text{sign}(x_{i_1} \dots x_{i_k}) = \text{sign}(x_{j_1} \dots x_{j_l}).$$

Now we repeat the second part in proof of Proposition 4. Let the signature $\text{sign}(\phi(f))$ has the highest weight λ , where $f \in \mathbb{C}[V]$. We look for a monomial $x_{i_1} \dots x_{i_k}$ such that the least term of $\phi(x_{i_1} \dots x_{i_k})$ coincides with the least term of $\phi(f)$. We can assume that f is a polynomial such that the least signature of a monomial in f is maximal over polynomials in $\phi^{-1}\phi(f)$. Let $h = \text{int}(f)$. If $h \notin J$, then we may take for $x_{i_1} \dots x_{i_k}$ any monomial of h . Otherwise $h \in J$, and since the first property holds we may assume that h is a linear combination of polynomials of the forms:

$$\underbrace{\dots}_{\text{monomial}} (x_{i_1}x_{i_2} - x_{j_1}x_{j_2}) \quad \text{or} \quad \underbrace{\dots}_{\text{monomial}} (x_{k_1}x_{k_2} - x_l),$$

where $x_{i_1}x_{i_2} - x_{j_1}x_{j_2}, x_{k_1}x_{k_2} - x_l \in J$.

Since the second property holds, hence for any binomial of the form

$$x_{i_1}x_{i_2} - x_{j_1}x_{j_2} \quad \text{or} \quad x_{k_1}x_{k_2} - x_l$$

there exists a polynomial g such that

$$\phi(g) = \phi(x_{i_1}x_{i_2} - x_{j_1}x_{j_2}) \quad \text{or} \quad \phi(g) = \phi(x_{k_1}x_{k_2} - x_l),$$

$$\text{sign}(\text{int}(g)) > \text{sign}(x_{i_1}x_{i_2} - x_{j_1}x_{j_2}) \quad \text{or} \quad \text{sign}(\text{int}(g)) > \text{sign}(x_{k_1}x_{k_2} - x_l).$$

After replacing all binomials by such polynomials g we obtain a polynomial h' such that $\phi(h) = \phi(h')$ and $\text{sign}(h') > \text{sign}(h)$. Let $f' = f - h + h'$. Then $\phi(f) = \phi(f')$ and $\text{sign}(\text{int}(f)) < \text{sign}(\text{int}(f'))$. A contradiction. \square

4 Orthogonal case

In this section we prove that for some fixed numeration of positive roots and monomial order the semigroup Σ is generated by essential signatures of highest weights ω_i and $2\omega_n$ for B_n , and by essential signatures of highest weights ω_i , $2\omega_{n-1}$, $2\omega_n$, $\omega_{n-1} + \omega_n$ for D_n . We show that if this is true for D_n then it is true for D_{n+1} and B_n as well.

Let $\widehat{\omega}_p = \omega_p$ if $p \neq n-1$, and $\widehat{\omega}_{n-1} = \omega_{n-1} + \omega_n$ for D_n , and let $\widehat{\omega}_p = \omega_p$ if $p \neq n$, and $\widehat{\omega}_n = 2\omega_n$ for B_n .

Let $\pm\varepsilon_i, i = 1, \dots, n$ be the nonzero weights of representation $V(\omega_1)$ of D_n or B_n . Then the set of positive roots for D_n is

$$\varepsilon_i \pm \varepsilon_j, \quad i < j; i, j \in \{1, \dots, n\},$$

and the set of positive roots for B_n is

$$\varepsilon_i \pm \varepsilon_j, \varepsilon_i, \quad i < j; i, j \in \{1, \dots, n\},$$

Denote by $\Sigma_X(\lambda)$ the set of essential signatures of highest weight λ for Lie algebra of type X (we consider signatures in $\Sigma_X(\lambda)$ without highest weight), $V_X(\lambda)$ stands for the representation of Lie algebra of type X with the highest weight λ .

4.1 From D_n to B_n

In this section we assume that we have some numeration of positive roots and some monomial order for D_n such that Conjecture 2 is true and the properties * and ** (see Theorem 1) hold, where

$$\{\lambda_i\} = \{\omega_1, \dots, \omega_n, 2\omega_{n-1}, 2\omega_n, \omega_{n-1} + \omega_n\}.$$

We prove such result for B_n , where

$$\{\lambda_i\} = \{\omega_1, \dots, \omega_n, 2\omega_n\}.$$

First of all we want to extend the monomial order and the numeration of positive roots from D_n to B_n . We extend the numeration as follows:

positive roots of $D_n, \varepsilon_1, \dots, \varepsilon_n$.

We extend the monomial order as follows:

$$(\sigma, k_1, \dots, k_n) < (\sigma', k'_1, \dots, k'_n),$$

if either $(k_1, \dots, k_n) < (k'_1, \dots, k'_n)$ in graded lexicographic order or $(k_1, \dots, k_n) = (k'_1, \dots, k'_n)$ and $\sigma < \sigma'$ in type D_n .

Now we describe the set of essential signatures of highest weights λ_i for B_n . We denote by $\bar{\alpha}_i$ the signature with coordinate 1, corresponding to root α_i , and with coordinate 0 otherwise. We omit the highest weight coordinate in this signature. We consider a signature σ of D_n as the signature $(\sigma, 0, \dots, 0)$ of B_n .

Lemma 2.

$$\Sigma_{B_n}(\omega_p) = \Sigma_{D_n}(\omega_p) \cup (\Sigma_{D_n}(\omega_{p-1}) + \bar{\varepsilon}_p), p \neq n-1,$$

$$\Sigma_{B_n}(\omega_{n-1}) = \Sigma_{D_n}(\omega_{n-1} + \omega_n) \cup (\Sigma_{D_n}(\omega_{n-2}) + \bar{\varepsilon}_{n-1}),$$

$$\Sigma_{B_n}(2\omega_n) = (\Sigma_{D_n}(\omega_{n-1} + \omega_n) + \bar{\varepsilon}_n) \cup (\Sigma_{D_n}(2\omega_{n-1}) + 2\bar{\varepsilon}_n) \cup \Sigma_{D_n}(2\omega_n).$$

$$\Sigma_{D_n}(\omega_0) := \{\sigma = (0, \dots, 0)\}.$$

Proof. Denote by $e_{\pm i}$ eigenvectors $V(\omega_1)$ of eigenvalues $\pm \varepsilon_i$, and denote by e_0 an eigenvector of eigenvalue 0 (for B_n). Then for any $p < n-1$ one has $V_{B_n}(\omega_p) = \bigwedge^p(\mathbb{C}^{2n+1})$, $V_{D_n}(\omega_p) = \bigwedge^p(\mathbb{C}^{2n})$, and $v_{\omega_p} = e_1 \wedge \dots \wedge e_p$ for D_n and B_n . The signatures $\Sigma_{D_n}(\omega_p)$ are signatures (for B_n) with zero coordinates, corresponding to roots ε_i , hence they are essential (see extended monomial order). One has $\langle v(\sigma) \mid \sigma \in \Sigma_{D_n}(\omega_p) \rangle = \langle e_{i_1} \wedge \dots \wedge e_{i_p} \mid i_j \in \pm\{1, \dots, n\} \rangle$. So to generate $V_{B_n}(\omega_p)$ we need vectors of the form $e_0 \wedge v$. The signatures $\Sigma_{D_n}(\omega_{p-1}) + \bar{\varepsilon}_p$ are minimal signatures such that they generate subspace $\langle e_0 \wedge e_{i_1} \wedge \dots \wedge e_{i_{p-1}} \mid i_j \in \pm\{1, \dots, n\} \rangle$. Therefore they are essential for B_n .

Same arguments show that

$$\Sigma_{B_n}(\omega_{n-1}) = \Sigma_{D_n}(\omega_{n-1} + \omega_n) \cup (\Sigma_{D_n}(\omega_{n-2}) + \bar{\varepsilon}_{n-1}).$$

Indeed, $V_{B_n}(\omega_{n-1}) = \bigwedge^{n-1} \mathbb{C}^{2n+1}$. The representation of D_n with the same highest weight vector in this case is exactly $V_{D_n}(\omega_{n-1} + \omega_n)$.

The equality

$$\Sigma_{B_n}(\omega_n) = \Sigma_{D_n}(\omega_n) \cup (\Sigma_{D_n}(\omega_{n-1}) + \bar{\varepsilon}_n)$$

is easy to verify since all weight subspaces of $V_{B_n}(\omega_n)$ are one-dimensional.

The last equality we have to prove is:

$$\Sigma_{B_n}(2\omega_n) = (\Sigma_{D_n}(\omega_{n-1} + \omega_n) + \bar{\varepsilon}_n) \cup (\Sigma_{D_n}(2\omega_{n-1}) + 2\bar{\varepsilon}_n) \cup \Sigma_{D_n}(2\omega_n).$$

$V_{B_n}(2\omega_n) = \bigwedge^n \mathbb{C}^{2n+1}$. It is easy to verify that $\Sigma_{D_n}(\omega_{n-1} + \omega_n) + \bar{\varepsilon}_n$ are minimal signatures which generate subspace $e_0 \wedge v$. $\Sigma_{D_n}(2\omega_{n-1}) + 2\bar{\varepsilon}_n$ and $\Sigma_{D_n}(2\omega_n)$ are essential since $V_{D_n}(2\omega_{n-1}) + V_{D_n}(2\omega_n) = \bigwedge^n \mathbb{C}^{2n}$. \square

Theorem 2. *The property * holds for B_n .*

Proof. By Lemma 2 we have the map (we just forget about coordinates, corresponding to roots ε_i):

$$\phi : \Sigma_{B_n}(\omega_p) \rightarrow \Sigma_{D_n}(\omega_p) \cup (\Sigma_{D_n}(\omega_{p-1})), p \neq n-1,$$

$$\phi : \Sigma_{B_n}(\omega_{n-1}) \rightarrow \Sigma_{D_n}(\omega_{n-1} + \omega_n) \cup (\Sigma_{D_n}(\omega_{n-2})),$$

$$\phi : \Sigma_{B_n}(2\omega_n) \rightarrow (\Sigma_{D_n}(\omega_{n-1} + \omega_n)) \cup (\Sigma_{D_n}(2\omega_{n-1})) \cup \Sigma_{D_n}(2\omega_n).$$

Assume that we have two decompositions of some signature σ of highest weight $\lambda = \sum k_i \omega_i$ in B_n :

$$\sigma_1 + \dots + \sigma_k = \tau_1 + \dots + \tau_l,$$

where σ_i and τ_j are essential signatures for B_n of highest weights $\omega_1, \dots, \omega_n, 2\omega_n$. Then we can apply the map ϕ to these decompositions, and obtain two signatures in D_n :

$$\phi(\sigma_1) + \dots + \phi(\sigma_k) \quad \phi(\tau_1) + \dots + \phi(\tau_l).$$

We claim that these two signatures coincide. Obviously, these two signatures have the same coordinates (see the map ϕ), hence we have to verify if the highest weights of this signatures coincide. Let t_i be the coordinates of σ , corresponding to roots ε_i respectively. It is easy to see that the highest weight of both signatures in D_n is $\sum k'_i \omega_i$, where

$$k'_i = k_i + t_{i+1} - t_i, \quad i < n,$$

$$k'_n = k_{n-1} + k_n - t_n - t_{n-1}.$$

Hence the highest weights coincide and

$$\phi(\sigma_1) + \dots + \phi(\sigma_k) = \phi(\tau_1) + \dots + \phi(\tau_l).$$

Remind that we can obtain right decomposition from the left by applying admissible operations in D_n since $*$ holds for D_n . Obviously, we can lift any admissible operation from D_n to B_n . Therefore we may assume that $k = l$ and:

$$\begin{aligned} \sigma_1 + \dots + \sigma_l &= \tau_1 + \dots + \tau_l, \\ \phi(\sigma_1) &= \phi(\tau_1) \quad \dots \quad \phi(\sigma_l) = \phi(\tau_l). \end{aligned}$$

Now the statement is obvious. \square

Theorem 3. *The property $**$ holds for B_n .*

Proof. We need to verify property $**$.

To prove property $**$ we will compute the number of signatures in $\Sigma_{B_n}(\lambda_p) + \Sigma_{B_n}(\lambda_q)$, and compare it with $\dim V_{B_n}(\lambda_p + \lambda_q)$ for all pairs of λ_p, λ_q except ω_n, ω_n .

Consider, for example, the pair ω_p, ω_q . By Lemma 2 we have the following equality:

$$\begin{aligned} |\Sigma_{B_n}(\omega_p) + \Sigma_{B_n}(\omega_q)| &= |\Sigma_{D_n}(\widehat{\omega}_p + \widehat{\omega}_q)| + |\Sigma_{D_n}(\omega_{p-1} + \widehat{\omega}_q)| + \\ &+ |\Sigma_{D_n}(\widehat{\omega}_p + \omega_{q-1})| + |\Sigma_{D_n}(\omega_{p-1} + \omega_{q-1})|, p \neq q; \\ |\Sigma_{B_n}(\omega_p) + \Sigma_{B_n}(\omega_p)| &= |\Sigma_{D_n}(\widehat{\omega}_p + \widehat{\omega}_p)| + |\Sigma_{D_n}(\omega_{p-1} + \widehat{\omega}_p)| + \\ &+ |\Sigma_{D_n}(\omega_{p-1} + \omega_{p-1})|. \end{aligned}$$

Hence we have to verify if the following equalities hold:

1. $\dim V_{B_n}(\omega_p + \omega_q) = \dim V_{D_n}(\widehat{\omega}_p + \widehat{\omega}_q) + \dim V_{D_n}(\widehat{\omega}_p + \omega_{q-1}) + \dim V_{D_n}(\omega_{p-1} + \widehat{\omega}_q) + \dim V_{D_n}(\omega_{p-1} + \omega_{q-1}), p \neq q;$
2. $\dim V_{B_n}(2\omega_p) = \dim V_{D_n}(2\widehat{\omega}_p) + \dim V_{D_n}(\widehat{\omega}_p + \omega_{p-1}) + \dim V_{D_n}(2\omega_{p-1}).$

By [11] we have the formulas for B_n :

$$\dim V_{B_n}(\widehat{\omega}_p + \widehat{\omega}_q) = \frac{(p-q+1)(2n-p-q+2)}{(p+1)(2n-p+2)} C_{2n+1}^p C_{2n+3}^q, q \leq p \leq n;$$

$$\dim V_{B_n}(\widehat{\omega}_p + \omega_n) = 2^n \frac{2n-2p+2}{2n-p+2} C_{2n+1}^p, p \leq n.$$

And we have similar formulas for D_n :

$$\dim V_{D_n}(\widehat{\omega}_p + \widehat{\omega}_q) = \frac{(p-q+1)(2n-p-q+1)}{(p+1)(2n-p+1)} C_{2n}^p C_{2n+2}^q, q \leq p \leq n;$$

$$\dim V_{D_n}(\widehat{\omega}_p + \omega_n) = 2^{n-1} \frac{2n-2p+1}{2n-p+1} C_{2n}^p, p \leq n;$$

$$\dim V_{D_n}(3\omega_n) = 2^n \frac{1}{n+1} C_{2n-1}^{n-1};$$

$$\dim V_{D_n}(4\omega_n) = \frac{2}{(n+1)(n+2)} C_{2n-1}^{n-1} C_{2n+1}^n;$$

$$\dim V_{D_n}(2\omega_n + \widehat{\omega}_p) = \frac{2(n-p+1)^2}{(n+1)(2n-p+2)} C_{2n-1}^{n-1} C_{2n+1}^p.$$

One can easily verify that all equalities above are identities. Moreover, similar equalities for the rest pairs of λ_i are identities as well. Therefore ** holds. \square

Theorem 4. *Conjecture 2 is true for B_n .*

Proof. We want to prove that semigroup Σ is saturated for B_n , if it is true for D_n . Let $\mu = a_1\sigma_1 + \dots + a_k\sigma_k$ be a signature of B_n in $\Sigma_{\mathbb{Q}}$ of highest weight $\lambda = \sum k_i\omega_i$, where σ_i are some essential signatures of highest weights $\omega_1, \dots, \omega_{n-1}, 2\omega_n$, and $a_i \in \mathbb{Q}$. Let t_i be the coordinates of μ , corresponding to roots ε_i respectively. We can assume that $\sigma_i \neq (\omega_1; 0, \dots, 0) + \overline{\varepsilon}_1$ for $i = 1, \dots, k$. Indeed, otherwise we would have $a_i \in \mathbb{Z}$, because this signature is the only one with nonzero coordinate, corresponding to root ε_1 . We apply map ϕ (see proof of Theorem 2) and obtain:

$$\mu' = a_1\phi(\sigma_1) + \dots + a_k\phi(\sigma_k).$$

where μ' is some signature for D_n . Since Conjecture 2 is true for D_n we get:

$$\mu' = n_1\tau_1 + \dots + n_s\tau_s,$$

where τ_i are essential signatures of highest weights $\omega_1, \dots, \omega_n, 2\omega_{n-1}, 2\omega_n, \widehat{\omega}_{n-1}$, for D_n , and $n_i \in \mathbb{Z}_{>0}$. Moreover, we can assume that among τ_i we have no pairs of signatures of the following highest weights:

$$\{\omega_n, \omega_{n-1}\}, \{2\omega_n, \omega_{n-1}\}, \{\omega_n, 2\omega_{n-1}\}, \{2\omega_n, 2\omega_{n-1}\}.$$

The signature μ of B_n and the signature μ' of D_n have the same coordinates except coordinates, corresponding to roots ε_i . By Lemma 2 we will lift the signatures τ_i to signatures for B_n as follows:

1. lift k_1 signatures of highest weight ω_1 to the signatures of highest weight ω_1 , and lift the rest (t_2) signatures of highest weight ω_1 to signatures of highest weight ω_2 ;
2. lift $k_2 - t_2$ signatures of highest weight ω_2 to the signatures of highest weight ω_2 to obtain exactly k_2 signatures of highest weight ω_2 , and lift the rest (t_3) signatures of highest weight ω_2 to signatures of highest weight ω_3 ;
3. for $p = 3, \dots, n-1$ lift $k_p - t_p$ signatures of highest weight $\widehat{\omega}_p$ to the signatures of highest weight ω_p to obtain exactly k_p signatures of highest weight ω_p , and lift the rest (t_{p+1}) signatures of highest weight $\widehat{\omega}_p$ to the signatures of highest weight $\widehat{\omega}_{p+1}$;
4. lift all signatures of highest weights $\omega_n, 2\omega_n$ or $\omega_{n-1}, 2\omega_{n-1}$ to signatures of highest weights $\omega_n, 2\omega_n$ respectively.

We lift μ' by lifting τ_i . Obviously,

$$\mu = n_1\phi^{-1}(\tau_1) + \dots + n_s\phi^{-1}(\tau_s).$$

Hence we obtain that $\mu \in \Sigma$. □

4.2 From D_{n-1} to D_n

In this section we assume that we have some numeration of positive roots and some monomial order for D_{n-1} such that the properties *, ** and Conjecture 2 hold. We prove such results for D_n .

First of all we want to extend the monomial order and the numeration of positive roots from D_{n-1} to D_n . We extend the numeration as follows:

$$\text{positive roots of } D_{n-1}, \varepsilon_1 - \varepsilon_n, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_1 + \varepsilon_n, \dots, \varepsilon_{n-1} + \varepsilon_n.$$

We extend the monomial order as follows:

$$(\sigma, k_1, \dots, k_n, l_1, \dots, l_n) < (\sigma', k'_1, \dots, k'_n, l'_1, \dots, l'_n),$$

if either $(l_1, \dots, l_n) < (l'_1, \dots, l'_n)$ in graded lexicographic order or $(l_1, \dots, l_n) = (l'_1, \dots, l'_n)$ and $(k_1, \dots, k_n) < (k'_1, \dots, k'_n)$ in graded lexicographic order or $(k_1, \dots, k_n, l_1, \dots, l_n) = (k'_1, \dots, k'_n, l'_1, \dots, l'_n)$ and $\sigma < \sigma'$ in type D_{n-1} .

Now we describe the set of essential signatures of highest weights λ_i for D_n .

Lemma 3.

$$\begin{aligned} \Sigma_{D_n}(\omega_p) &= \Sigma_{D_{n-1}}(\widehat{\omega}_p) \cup (\Sigma_{D_{n-1}}(\omega_{p-1}) + (\overline{\varepsilon_p - \varepsilon_n})) \cup \\ &\cup (\Sigma_{D_{n-1}}(\omega_{p-1}) + (\overline{\varepsilon_p + \varepsilon_n})) \cup (\Sigma_{D_{n-1}}(\omega_{p-2}) + (\overline{\varepsilon_p + \varepsilon_n}) + (\overline{\varepsilon_{p-1} - \varepsilon_n})), p < n-1; \end{aligned}$$

$$\Sigma_{D_n}(\omega_{n-1}) = \Sigma_{D_{n-1}}(\omega_{n-1}) \cup (\Sigma_{D_{n-1}}(\omega_{n-2}) + (\overline{\varepsilon_{n-1} - \varepsilon_n})),$$

$$\Sigma_{D_n}(\omega_n) = \Sigma_{D_{n-1}}(\omega_{n-1}) \cup (\Sigma_{D_{n-1}}(\omega_{n-2}) + (\overline{\varepsilon_{n-1} + \varepsilon_n})),$$

$$\begin{aligned} \Sigma_{D_n}(2\omega_{n-1}) &= (\Sigma_{D_{n-1}}(\omega_{n-1} + \omega_{n-2}) + (\overline{\varepsilon_{n-1} - \varepsilon_n})) \cup \\ &\cup (\Sigma_{D_{n-1}}(2\omega_{n-2}) + 2(\overline{\varepsilon_{n-1} - \varepsilon_n})) \cup \Sigma_{D_{n-1}}(2\omega_{n-1}), \end{aligned}$$

$$\begin{aligned} \Sigma_{D_n}(2\omega_n) &= (\Sigma_{D_{n-1}}(\omega_{n-1} + \omega_{n-2}) + (\overline{\varepsilon_{n-1} + \varepsilon_n})) \cup \\ &\cup (\Sigma_{D_{n-1}}(2\omega_{n-2}) + 2(\overline{\varepsilon_{n-1} + \varepsilon_n})) \cup \Sigma_{D_{n-1}}(2\omega_{n-1}), \end{aligned}$$

$$\begin{aligned} \Sigma_{D_n}(\widehat{\omega}_{n-1}) &= \Sigma_{D_{n-1}}(2\omega_{n-1}) \cup (\Sigma_{D_{n-1}}(2\omega_{n-2}) + (\overline{\varepsilon_{n-1} - \varepsilon_n}) + (\overline{\varepsilon_{n-1} + \varepsilon_n})) \cup \\ &\cup (\Sigma_{D_{n-1}}(\widehat{\omega}_{n-2}) + (\overline{\varepsilon_{n-1} - \varepsilon_n})) \cup (\Sigma_{D_{n-1}}(\widehat{\omega}_{n-2}) + (\overline{\varepsilon_{n-1} + \varepsilon_n})) \cup \\ &\cup (\Sigma_{D_{n-1}}(\omega_{n-3}) + (\overline{\varepsilon_{n-1} + \varepsilon_n}) + (\overline{\varepsilon_{n-2} - \varepsilon_n})). \end{aligned}$$

$$\Sigma_{D_{n-1}}(\omega_0) := \{\sigma = (0, \dots, 0)\},$$

$$\Sigma_{D_{n-1}}(\omega_{-1}) := \{\sigma = (0, \dots, 0)\},$$

$$\overline{\varepsilon_0 - \varepsilon_n} := \{\sigma = (0, \dots, 0)\}.$$

Proof. We may extend the numeration and the monomial order from D_{n-1} to B_{n-1} . Denote by $\Sigma'_{B_{n-1}}(\lambda)$ ($\Sigma''_{B_{n-1}}(\lambda)$) the signatures from $\Sigma_{B_{n-1}}(\lambda)$, where coordinates corresponding to roots ε_i are replaced by coordinates corresponding to roots $\varepsilon_i - \varepsilon_n$ ($\varepsilon_i + \varepsilon_n$). Then one has:

$$\Sigma_{D_n}(\omega_p) = \Sigma'_{B_{n-1}}(\omega_p) \cup (\Sigma'_{B_{n-1}}(\omega_{p-1}) + (\overline{\varepsilon_p + \varepsilon_n})); p < n - 1,$$

$$\Sigma_{D_n}(\omega_{n-1}) = \Sigma'_{B_{n-1}}(\omega_{n-1}),$$

$$\Sigma_{D_n}(\omega_n) = \Sigma''_{B_{n-1}}(\omega_{n-1}),$$

$$\Sigma_{D_n}(2\omega_{n-1}) = \Sigma'_{B_{n-1}}(2\omega_{n-1}),$$

$$\Sigma_{D_n}(2\omega_n) = \Sigma''_{B_{n-1}}(2\omega_{n-1}).$$

The first equality can be proved as in Lemma 2. The rest equalities are obvious. Rewriting of these equalities in terms of signatures for D_{n-1} prove all equalities except the last one. The last equality is similar to the last equality from Lemma 2. It is easy to see that the signatures from $\Sigma_{D_{n-1}}(\widehat{\omega}_{n-2}) + (\overline{\varepsilon_{n-1} \pm \varepsilon_n})$ are minimal signatures which generate subspace of the form $e_{\pm n} \wedge v$, signatures from $\Sigma_{D_{n-1}}(\omega_{n-3}) + (\overline{\varepsilon_{n-1} + \varepsilon_n}) + (\overline{\varepsilon_{n-2} - \varepsilon_n})$ generate subspace $e_n \wedge e_{-n} \wedge v$, and signatures from $\Sigma_{D_{n-1}}(2\omega_{n-1}) \cup (\Sigma_{D_{n-1}}(2\omega_{n-2}) + (\overline{\varepsilon_{n-1} - \varepsilon_n}) + (\overline{\varepsilon_{n-1} + \varepsilon_n}))$ generate subspace $\langle e_{i_1} \wedge \dots \wedge e_{i_{n-1}} \mid i_j \in \pm\{1, \dots, n-1\} \rangle$. \square

Theorem 5. *The property $*$ holds for D_n .*

Proof. We act as in Theorem 2. By Lemma 3 we have the map ϕ (we just forget about coordinates, corresponding to roots $\varepsilon_i \pm \varepsilon_n$).

Assume that we have two decompositions of some signature σ of highest weight $\lambda = \sum k_i \omega_i$ in D_n :

$$\sigma_1 + \dots + \sigma_k = \tau_1 + \dots + \tau_l,$$

where σ_i and τ_j are essential signatures for D_n of highest weights $\omega_1, \dots, \omega_n, 2\omega_{n-1}, 2\omega_n, \omega_{n-1} + \omega_n$. Then we can apply the map ϕ to these decompositions, and obtain two signatures in D_{n-1} :

$$\phi(\sigma_1) + \dots + \phi(\sigma_k) \quad \phi(\tau_1) + \dots + \phi(\tau_l).$$

We claim that these two signatures coincide. Obviously, these two signatures have the same coordinates (see the map ϕ), hence we have to verify if the highest weights of this signatures coincide. Let t'_i (t''_i) be the coordinates of

σ , corresponding to roots $\varepsilon_i - \varepsilon_n$ ($\varepsilon_i + \varepsilon_n$) respectively. We may assume that $t'_1, t''_1 = 0$. It is easy to see that the highest weight of both signatures in D_{n-1} is $\sum k'_i \omega_i$, where

$$\begin{aligned} k'_i &= k_i + t'_{i+1} + t''_{i+1} - t'_i - t''_i, \quad i < n-1, \\ k'_{n-1} &= k_{n-2} + k_{n-1} + k_n - t'_{n-1} - t''_{n-1} - t'_{n-2} - t''_{n-2}. \end{aligned}$$

Hence the highest weights coincide and

$$\phi(\sigma_1) + \dots + \phi(\sigma_k) = \phi(\tau_1) + \dots + \phi(\tau_l).$$

We finish the proof acting as in Theorem 2. □

Theorem 6. *The property ** holds for D_n .*

Proof. Similar arguments as in Theorem 3. □

Theorem 7. *Conjecture 2 is true for D_n .*

Proof. We act as in Theorem 4, but the proof is technically complicated. We want to prove that semigroup Σ is saturated for D_n , if it is true for D_{n-1} . Let $\mu = a_1 \sigma_1 + \dots + a_k \sigma_k$ be a signature of D_n in $\Sigma_{\mathbb{Q}}$ of highest weight $\lambda = \sum k_i \omega_i$, where σ_i are some essential signatures of highest weights $\omega_1, \dots, \omega_{n-2}, 2\omega_{n-1}, 2\omega_n, \omega_{n-1} + \omega_n$ and $a_i \in \mathbb{Q}$. We can assume that $k_{n-1} \leq k_n$. Let t'_i (t''_i) be the coordinates of μ , corresponding to roots $\varepsilon_i - \varepsilon_n$ ($\varepsilon_i + \varepsilon_n$) respectively. It is easy to see that we can assume $t''_1, t'_1 = 0$. We apply map ϕ (see proof of Theorem 5) and obtain:

$$\mu' = a_1 \phi(\sigma_1) + \dots + a_k \phi(\sigma_k).$$

where μ' is some signature of highest weight $\sum k'_i \omega_i$ for D_{n-1} . Since Conjecture 2 is true for D_{n-1} we get:

$$\mu' = n_1 \tau_1 + \dots + n_s \tau_s,$$

where τ_i are essential signatures of highest weights $\omega_1, \dots, \omega_{n-1}, 2\omega_{n-2}, 2\omega_{n-1}, \widehat{\omega}_{n-2}$, for D_{n-1} , and $n_i \in \mathbb{Z}_{>0}$. Moreover, we can assume that $k'_{n-1} > k'_{n-2}$ and among τ_i we have no signatures of the following highest weights:

$$\omega_{n-2}, 2\omega_{n-2}.$$

The signature μ of D_n and the signature μ' of D_{n-1} have the same coordinates except coordinates, corresponding to roots $\varepsilon_i \pm \varepsilon_n$. By Lemma 3 we will lift the signatures τ_i to signatures for D_n as follows:

1. lift k_1 signatures of highest weight ω_1 to the signatures of highest weight ω_1 . Lift t_2'' signatures of highest weight ω_1 to signatures of highest weight ω_2 by adding $\overline{\varepsilon_2 + \varepsilon_n}$. If $t_2' > t_3''$ lift $t_2' - t_3''$ signatures of highest weight ω_1 to signatures of highest weight ω_2 by adding $\overline{\varepsilon_2 - \varepsilon_n}$ and lift t_3'' signatures of highest weight ω_1 to signatures of highest weight ω_3 ; If $t_2' \leq t_3''$ lift t_2' signatures of highest weight ω_1 to signatures of highest weight ω_3 and lift $t_3'' - t_2'$ signatures of highest weight ω_2 to signatures of highest weight ω_3 ;
2. lift some signatures of highest weight ω_2 to the signatures of highest weight ω_2 to obtain exactly k_2 signatures of highest weight ω_2 . If $t_3' > t_4''$ lift $t_3' - t_4''$ signatures of highest weight ω_2 to signatures of highest weight ω_3 by adding $\overline{\varepsilon_3 - \varepsilon_n}$ and lift t_4'' signatures of highest weight ω_2 to signatures of highest weight ω_4 ; If $t_3' \leq t_4''$ lift t_3' signatures of highest weight ω_2 to signatures of highest weight ω_4 and lift $t_4'' - t_3'$ signatures of highest weight ω_3 to signatures of highest weight ω_4 ;
3. for $p = 3, \dots, n-4$ lift some signatures of highest weight ω_p to the signatures of highest weight ω_p to obtain exactly k_p signatures of highest weight ω_p . If $t_{p+1}' > t_{p+2}''$ lift $t_{p+1}' - t_{p+2}''$ signatures of highest weight ω_p to signatures of highest weight ω_{p+1} by adding $\overline{\varepsilon_{p+1} - \varepsilon_n}$ and lift t_{p+2}'' signatures of highest weight ω_p to signatures of highest weight $\widehat{\omega}_{p+2}$; If $t_{p+1}' \leq t_{p+2}''$ lift t_{p+2}'' signatures of highest weight ω_p to signatures of highest weight $\widehat{\omega}_{p+2}$ and lift $t_{p+2}'' - t_{p+1}'$ signatures of highest weight $\widehat{\omega}_{p+1}$ to signatures of highest weight $\widehat{\omega}_{p+2}$;
4. for $p = n-3$ lift some signatures of highest weight ω_{n-3} to the signatures of highest weight ω_{n-3} to obtain exactly k_{n-3} signatures of highest weight ω_{n-3} . If $k_{n-2}' < t_{n-1}' + t_{n-1}'' - k_{n-2}'$ lift $t_{n-1}' + t_{n-1}'' - k_{n-2}'$ signatures of highest weight ω_{n-3} to signatures of highest weight $\widehat{\omega}_{n-1}$. Lift some signatures of highest weight ω_{n-3} to signatures of highest weight ω_{n-2} by adding $\overline{\varepsilon_{n-2} - \varepsilon_n}$ to obtain exactly t_{n-2}' signatures with nonzero coordinate corresponding to root $\varepsilon_{n-2} - \varepsilon_n$;
5. lift some signatures of highest weight $\widehat{\omega}_{n-2}$ to obtain k_{n-2} signatures of highest weight ω_{n-2} ;
6. lift all signatures of highest weights ω_{n-1} to signatures of highest weights ω_n , and lift signatures of highest weight $2\omega_{n-1}, \widehat{\omega}_{n-2}$ to signatures of

highest weight $\widehat{\omega}_{n-1}, 2\omega_n$ to obtain exactly t'_{n-1} (t''_{n-1}) signatures with nonzero coordinate corresponding to root $\varepsilon_{n-1} - \varepsilon_n$ ($\varepsilon_{n-1} + \varepsilon_n$).

We lift μ' by lifting τ_i . Obviously,

$$\mu = n_1\phi^{-1}(\tau_1) + \dots + n_s\phi^{-1}(\tau_s).$$

Hence we obtain that $\mu \in \Sigma$. □

4.3 D_3

Now we want to prove that we can start inductive procedure with D_3 . We show that properties from Proposition 5 hold for some numeration and monomial order for D_3 .

Consider the case D_3 . Let $\beta_1, \beta_2, \beta_3$ be the simple roots for D_3 . One has:

$$\begin{aligned} \beta_1 &= \varepsilon_1 - \varepsilon_2 & \beta_2 &= \varepsilon_2 - \varepsilon_3 & \beta_3 &= \varepsilon_2 + \varepsilon_3. \\ \omega_1 &= \varepsilon_1 & \omega_2 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3) & \omega_3 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3). \end{aligned}$$

Let us number the positive roots of D_3 as follows:

$$\begin{aligned} \alpha_1 &= \varepsilon_2 - \varepsilon_3 & \alpha_2 &= \varepsilon_1 - \varepsilon_3 \\ \alpha_3 &= \varepsilon_1 + \varepsilon_2 & \alpha_4 &= \varepsilon_1 - \varepsilon_2 \\ \alpha_5 &= \varepsilon_1 + \varepsilon_3 & \alpha_6 &= \varepsilon_2 + \varepsilon_3 \end{aligned}$$

Let $\sigma = (\lambda; p_1, \dots, p_6)$, $\sigma' = (\lambda; p'_1, \dots, p'_6)$. We say $\sigma > \sigma'$ if $(p_1, \dots, p_6) > (p'_1, \dots, p'_6)$ in graded lexicographic order.

Here are all essential signatures of the highest weight ω_1 (the highest weight component is omitted):

1. $(0, 0, 0, 0, 0, 0)$ 2. $(0, 1, 0, 0, 0, 0)$ 3. $(0, 0, 1, 0, 0, 0)$
4. $(0, 0, 0, 1, 0, 0)$ 5. $(0, 0, 0, 0, 1, 0)$ 6. $(0, 0, 1, 1, 0, 0)$

Here are all essential signatures of the highest weight ω_2 (the highest weight component is omitted):

1. $(0, 0, 0, 0, 0, 0)$ 2. $(1, 0, 0, 0, 0, 0)$
3. $(0, 1, 0, 0, 0, 0)$ 4. $(0, 0, 1, 0, 0, 0)$

Here are all essential signatures of the highest weight ω_3 (the highest weight component is omitted):

1. $(0, 0, 0, 0, 0, 0)$ 2. $(0, 0, 1, 0, 0, 0)$
3. $(0, 0, 0, 0, 1, 0)$ 4. $(0, 0, 0, 0, 0, 1)$

Theorem 8. *The ideal J is generated by binomials of degree 2.*

Proof. The proof is obvious. □

Notice that this numeration and monomial order are the same as in [2] for $A_3 = D_3$. Therefore, Conjecture 1 and Conjecture 2 are true. Hence properties from Proposition 5 (and hence properties * and **) hold for D_3 . So we can start inductive procedure from D_3 .

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